EPFL

Do not turn this page until instructed

Probability and Statistics for IC
2022, Spring semester

Probability and Statistics: Exam

June 20, 2022

Duration: The exam will start at 15:15 and end at 18:15 (unless special arrangements).

Family name:

First name:

SCIPER number:

Exercise	Points	Indicative marks
1		/4 points
2		/2 points
3		/3 points
4		/3 points
5		/3 points
6		/7 points
7		/3 points
Total:		/25 points

PROTOCOL:

- If caught cheating, you will get a 0 and we will report to the section.
- No personal documents, cheat sheets or calculators are allowed during the exam.
- Justify all your answers! Unjustified answers will not get full score even if correct. However, partial reasoning might get partial points.
- Try to simplify numerical expressions but no need to give exact decimal expressions (e.g., you can leave factorial expressions).
- Ask a question only if it is crucial. There should be no assistance needed with the exam questions, so if your question is unnecessary, we will not reply to it.
- There are 7 problems. After each problem, there is some blank space for the solution. If you need to add extra pages, please write your name on all sheets you add and make sure they are stapled together with the rest of the exam when collected.

Grading Scheme

Exercise 1.

1.-2. (1.5 pt)

3. (0.5pt)
4. (1pt)
5. (1pt)

Exercise 2.

1. (1pt)
2. (1pt)

Exercise 3.

1. (1pt)
2. (1pt)
3. (1pt)

Exercise 4.

1. (1pt)
2. (2pt)
   a) (1.5pt)
   b) (0.5pt)

Exercise 5.

1. (1.5pt)
2. (1.5pt)
   a) (0.75pt)
   b) (0.75pt)

Exercise 6.

1. (2pt)
   a) (1pt)
   b) (1pt)
2. (1pt)
3. (1pt)
4. (3pt)
   a) (1pt)
   b) (1pt)
   c) (1pt)

Exercise 7.

1. (2pt)
2. (1pt)

Solutions

Exercise 1. We consider 5 urns: 3 of type A and 2 of type B. Each urn of type A contains 1 red ball and 3 white balls; each urn of type B contains 2 red balls and 2 white balls. We sample an urn uniformly at random; then, in the chosen urn, we sample two balls uniformly at random, *with replacement* (i.e., we replace the first sampled ball back in the urn before sampling the second ball).
Let U_A (resp. U_B) denote the event "the chosen urn is of type A (resp. of type B)". Let X denote number of red balls among the sampled balls.

1. Compute $\mathbb{P}(U_A)$, $\mathbb{P}(U_B)$ and $\mathbb{P}(X = k|U_B)$ for $k = 0, 1, 2$.

2. What is the probability mass function of X?

3. Compute $\mathbb{E}[X]$.

4. Knowing that we have sampled a red ball and a white ball, what is the probability that we have sampled an urn of type A?

5. Are the events $\{X = 1\}$ and U_A independent? As always, justify your answer.

Solution:

1.

$$\begin{aligned}\mathbb{P}(U_A) &= \frac{3}{5}\\ \mathbb{P}(U_B) &= \frac{2}{5}\\ \mathbb{P}(X = k|U_B) &= \binom{2}{k} \cdot \frac{1}{4}\end{aligned}$$

2.

$$\begin{aligned}\mathbb{P}(X = k) &= \mathbb{P}(X = k|U_A)\mathbb{P}(U_A) + \mathbb{P}(X = k|U_B)\mathbb{P}(U_B)\\ &= \frac{3}{5}\binom{2}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{2-k} + \frac{2}{5}\binom{2}{k}\frac{1}{4}\end{aligned}$$

3. One can observe that $X|U_A \sim \text{Bin}(1/4, 2)$ and $X|U_B \sim \text{Bin}(1/2, 2)$. Thus, $\mathbb{E}[X|U_A] = 2/3$ and $\mathbb{E}[X|U_B] = 1$. We can compute

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[X|U_A] \cdot \mathbb{P}(U_A) + \mathbb{E}[X|U_B] \cdot \mathbb{P}(U_B)\\ &= \frac{1}{2} \cdot \frac{3}{5} + 1 \cdot \frac{2}{5} = \frac{7}{10}.\end{aligned}$$

4. We first compute

$$\begin{aligned}\mathbb{P}(X = 1) &= \mathbb{P}(X = 1|U_A)\mathbb{P}(U_A) + \mathbb{P}(X = 1|U_B)\mathbb{P}(U_B)\\ &= 2 \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{5} + 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{5} = \frac{17}{40}\end{aligned}$$

We then apply Bayes-rule

$$\begin{aligned}\mathbb{P}(U_A|X = 1) &= \frac{\mathbb{P}(X = 1|U_A) \cdot \mathbb{P}(U_A)}{\mathbb{P}(X = 1)}\\ &= \frac{3/8 \cdot 3/5}{17/40} = \frac{9}{17}\end{aligned}$$

5. No. In fact, $\mathbb{P}(X = 1) \neq \mathbb{P}(X = 1|U_A)$.

Exercise 2. Let $(X,Y) \in \mathbb{R}^2$ be a random point, sampled uniformly in the unit disk. Said differently, (X,Y) has density

$$f_{(X,Y)}(x,y) = \frac{1}{\pi} I(x^2+y^2 \leq 1) = \begin{cases} \frac{1}{\pi} & \text{if } x^2+y^2 \leq 1\,, \\ 0 & \text{if } x^2+y^2 > 1\,. \end{cases}$$

1. What is the density of X?

2. Are X and Y independent? (As always, justify your answer.)

Solution:

1.

$$\begin{aligned} f_X(x) &= \int_{\mathbb{R}} f_{(X,Y)}(x,y)dy \\ &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \\ &= \frac{2}{\pi}\sqrt{1-x^2} \end{aligned}$$

Note: Another correct approach is the graphical solution yielding $\frac{2}{\pi}\sin(\arccos(x))$.

2. No. In fact, by symmetry, $f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}$. Take for example, $(x,y) = (5/7, 5/7)$. Note that $(5/7)^2 + (5/7)^2 = \frac{50}{49} > 1$.

$$\begin{aligned} f_{(X,Y)}(5/7,5/7) &= 0 \\ f_X(5/7)f_Y(5/7) &= \frac{4}{\pi^2}(1-25/49) > 0. \end{aligned}$$

Since there exist (x,y) such that $f_{(X,Y)}(x,y) \neq f_X(x)f_Y(y)$, it follows that X and Y are not independent.

Exercise 3. Let U_1, U_2, U_3 be independent identically distributed (i.i.d.) $\mathcal{N}(0,1)$ random variables.

1. What is the distribution of $X = \begin{pmatrix} U_1 - 2U_2 \\ U_1 + U_2 + U_3 \\ U_2 - U_3 \end{pmatrix}$?

2. Are the random variables $Y = U_1 + U_2 + U_3$ and $Z = U_2 - U_3$ independent?

3. Show that the random variable $T = \frac{Y^2}{3} + \frac{Z^2}{2}$ has a chi-square distribution χ^2_2 with 2 degrees of freedom.

Solution:

1. Observe that $X = AU$, where

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \qquad \text{and} \qquad U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}$$

Since U_1, U_2, U_3 are independent and Gaussians, then they are jointly Gaussian and $U \sim \mathcal{N}_3(0, I_3)$. Thus, X is a linear transformation of a Gaussian vector, which means that it is itself Gaussian, specifically $X \sim \mathcal{N}_3(0, AA^T)$, where

$$AA^T = \begin{pmatrix} 5 & -1 & -2 \\ -1 & 3 & 0 \\ -2 & 0 & 2 \end{pmatrix}.$$

(One can also compute manually the variance-covariance matrix of X).

2. From point 1, it follows that Y and Z are jointly Gaussian and uncorrelated. Thus, they are independent.

3. Let $\tilde{Y} = Y/\sqrt{3}$ and $\tilde{Z} = Z/\sqrt{2}$. Since $Y \sim \mathcal{N}(0,3)$ and $Z \sim (0,2)$, it follows that $\tilde{Y}, \tilde{Z} \sim \mathcal{N}(0,1)$. Since Y and Z are independent, also $\tilde{Y}$ and $\tilde{Z}$ are independent. Thus,

$$T = \tilde{Z}^2 + \tilde{Y}^2 \sim \chi^2_2.$$

For the following exercise, we recall this proposition:

Proposition 1. *Let X be a random variable with density f_X and $g : \mathbb{R} \to \mathbb{R}$ be a monotone increasing (or decreasing) map, with differentiable inverse g^{-1}. Then $Y = g(X)$ has density*

$$f_Y(y) = \left| \frac{\mathrm{d}g^{-1}(y)}{\mathrm{d}y} \right| f_X(g^{-1}(y)) \,.$$

Exercise 4. Let X be a random variable with density

$$f(x;\alpha) = \frac{2}{\alpha} x \exp\left(-\frac{x^2}{\alpha}\right) I(x>0)\,,$$

where $I(x>0)$ denotes the indicator of $x>0$, and $\alpha > 0$ is a parameter of the distribution.

1. What is the distribution of X^2?

2. We observe an independent identically distributed (i.i.d.) sample $(x_1, \dots, x_n)$ from $f(\,.\,;\alpha)$, but α is unknown.

(a) Give the maximum likelihood estimate $\hat{\alpha}$ for α.

(b) Is $\hat{\alpha}$ unbiased in estimating α?

Solution:

1. We apply Proposition 1 with $g : (0,+\infty) \to (0,\infty)$ given by $g(x) = x^2$, $g^{-1}(y) = \sqrt{y}$ and $dg^{-1}(y) = \frac{1}{2\sqrt{y}}$. One can check that g is monotonically increasing on its domain and g^{-1} is differentiable. By Proposition 1 we get

$$f_Y(y;\alpha) = \frac{1}{\alpha}\exp(-y/\alpha) I(y>0), \tag{1}$$

which implies that $Y \sim \mathrm{Exp}(1/\alpha)$.

2. a) Let us consider $(y_1, ..., y_n) = (x_1^2, ..., x_n^2)$. By construction, $(y_1, ..., y_n)$ is an i.i.d. sample from $f_Y(y;\alpha)$. We compute the maximum likelihood estimate using the y_i. In fact, the log-likelihood of α after observing $(y_1, ..., y_n)$ is given by

$$\begin{aligned} \ell(\alpha) &= \sum_{i=1}^n \log(1/\alpha^n) - y_i/\alpha \\ &= -n(\log(\alpha) - \bar{y}/\alpha), \end{aligned}$$

where we denoted $\bar{y} = \frac{1}{n}\sum_{i=1}^n y_i$. We compute the max-likelihood estimator by

$$\frac{\partial \ell(\alpha)}{\partial \alpha} = -n\left(\frac{1}{\alpha} - \frac{\bar{y}}{\alpha^2}\right) = 0 \iff \alpha = \bar{y}.$$

One should check that $\frac{\partial^2 \ell(\alpha)}{\partial \alpha^2}\, |_{\alpha=\bar{y}} < 0$.

b) Yes, it is unbiased:

$$\mathbb{E}[\hat{\alpha}] = \mathbb{E}[\bar{y}] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[y_i] = \alpha.$$

Exercise 5. We perform ten blood tests on a single individual. For each one of them, we obtain the following cholesterol levels (in grams):

245, 248, 247, 247, 249, 247, 247, 246, 246, 248.

We model this sequence of measurements as independent realizations of a random variable X "cholesterol level" whose distribution is assumed to be normal with mean μ and variance σ^2.
In this exercise, we seek to build confidence intervals using *exact* (not asymptotic) pivots for the parameters μ and σ^2. Try to give confidence intervals in terms of numerical expressions, but no need to simplify those (e.g. you can leave fractions, square roots, etc..). Please, write the full derivation of the confidence intervals asked; formulas with no derivation might not get full score, even if correct.

1. Assume first that σ^2 is known, and equal to 1.5. Build an equi-tailed confidence interval for μ, with confidence level 95%.

2. Assume now that σ^2 is unknown.

(a) Build an equi-tailed confidence interval for μ, with confidence level 95%.

(b) Build an equi-tailed confidence interval for σ^2, with confidence level 95%.

Solution:

1. If σ^2 is known, then

$$Z = \frac{\bar{X}-\mu}{\sqrt{\sigma^2/n}} \sim \mathcal{N}(0,1)$$

is a pivot, where we denoted by $n = 10$ the total number of observations and by $\bar{X} = 247$ the sample mean. This provides exact $(1-\alpha_L-\alpha_U)$ confidence interval for μ of the form

$$(L,U) = \left(\bar{X} - \frac{\sigma}{\sqrt{n}} z_{1-\alpha_L}, \bar{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha_U}\right),$$

where z_p denotes the p quantile of the standard normal distribution. By setting $\alpha_U = \alpha_L = 0.025$ and taking the relevant numbers from table of normal cumulative distribution function we get

$$(L,U) = \left(247 - \sqrt{0.15}\cdot 1.96, 247 + \sqrt{0.15}\cdot 1.96\right).$$

2. a) Let $S^2 := \frac{1}{n-1}\sum_{j=1}^{n}(X_i-\bar{X})^2 = 12/9 = 4/3$ denote the sample variance. Then,

$$T := \frac{\bar{X}-\mu}{\sqrt{S^2/n}} = Z \cdot \frac{1}{\sqrt{S^2/\sigma^2}} \sim t_{n-1},$$

since $Z \sim \mathcal{N}(0,1)$ and $S^2/\sigma^2 \sim \chi^2_{n-1}/(n-1)$ are independent (this follows from Theorem 274 stated in the class). Thus, T is a pivot that provides that following confidence interval from μ

$$\begin{aligned}(L,U) &= \left(\bar{X} - \frac{S}{\sqrt{n}} t_{n-1}(1-\alpha_L), \bar{X} + \frac{S}{\sqrt{n}} t_{n-1}(\alpha_U)\right)\\ &= \left(247 - \sqrt{\frac{52}{90}}\cdot 2.262, 247 + \sqrt{\frac{52}{90}}\cdot 2.262\right)\end{aligned}$$

b)

$$C := \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

is a pivot that provides the following confidence intervals for σ^2:

$$\begin{aligned}(L, U) &= \left(\frac{(n-1)S^2}{\chi^2_{n-1}(1-\alpha_L)}, \frac{(n-1)S^2}{\chi^2_{n-1}(\alpha_U)}\right) \\ &= \left(\frac{52}{19.023}, \frac{52}{2.7}\right)\end{aligned}$$

For the following exercise, you can use this fact:

Fact 1. *Let $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ be two sequences of random variables and let x, y be two real numbers. If $X_n \xrightarrow[n\to\infty]{} x$ almost surely and $Y_n \xrightarrow[n\to\infty]{} y$ almost surely, then $X_n + Y_n \xrightarrow[n\to\infty]{} x + y$ almost surely.*

Exercise 6. Let $X_1, X_2, X_3, \ldots$ be an infinite sequence of independent identically distributed (i.i.d.) Ber$(1/2)$ random variables.

1. Define $I_{1,n}$ the number of ones in the sequence $X_1, \ldots, X_n$, i.e.,

$$I_{1,n} = \#\{1 \le i \le n \,|\, X_i = 1\}\,.$$

(a) Give a number b_1 such that $\frac{1}{n}I_{1,n} \xrightarrow[n\to\infty]{} b_1$ almost surely. Justify your answer.

(b) Give a sequence $(c_n)_{n\ge1}$ of real numbers, and a real number d, such that

$$c_n\left(\frac{1}{n}I_{1,n} - d\right) \xrightarrow[n\to\infty]{} Z \quad \text{in distribution,}$$

where Z is a standard Gaussian random variable. Justify your answer.

2. Define $I_{2,n}$ the number of pairs of consecutive ones in the sequence $X_1, \ldots, X_n$, i.e.,

$$I_{2,n} = \#\{1 \le i \le n-1 \,|\, X_i = X_{i+1} = 1\}\,.$$

Give a number b_2 such that $\frac{1}{n}I_{2,n} \xrightarrow[n\to\infty]{} b_2$ almost surely. Justify your answer.

3. Define $I_{k,n}$ the number of sub-sequences of k consecutive ones in the sequence $X_1, \ldots, X_n$, i.e.,

$$I_{k,n} = \#\{1 \le i \le n-k+1 \,|\, X_i = X_{i+1} = \cdots = X_{i+k-1} = 1\}\,.$$

Give a number b_k such that $\frac{1}{n}I_{k,n} \xrightarrow[n\to\infty]{} b_k$ almost surely. Justify your answer.

4. Let K_n be the length of the largest sub-sequence of ones in the sequence $X_1, \ldots, X_n$, i.e.,

$$K_n = \max\{k \in \mathbb{N} \,|\, I_{k,n} > 0\}\,.$$

(By convention, we assume $I_{0,n} = n$ so that K_n is well defined.)

(a) Let k, n be two positive integers such that $k \le n$. Show that $\mathbb{P}(K_n \ge k) \le \frac{n}{2^k}$.

(b) Let k, n be two positive integers such that $k \le n$. Show that $\mathbb{P}(K_n < k) \le \left(1 - \frac{1}{2^k}\right)^{\lfloor n/k \rfloor}$, where $\lfloor n/k \rfloor$ denotes the integer part of n/k.

(c) Using the two questions above, show that $\frac{\log 2}{\log n}K_n \xrightarrow[n\to\infty]{} 1$ in probability.

Solution:

1. a) We observe that $\frac{1}{n}I_{1,n} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$. By the strong law of large numbers,

$$\bar{X} \xrightarrow[n\to\infty]{a.s.} \mathbb{E}[X_1] = \frac{1}{2}.$$

   Thus, $b_1 = \frac{1}{2}$.

   b) By central limit theorem,

$$\frac{\bar{X} - \mathbb{E}[\bar{X}]}{\sqrt{\mathrm{Var}(\bar{X})}} = \frac{\bar{X} - \frac{1}{2}}{\sqrt{1/4n}} \sim \mathcal{N}(0,1).$$

   Thus, $c_n = 2\sqrt{n}$ and $d = \frac{1}{2}$.

2. One can write

$$\frac{1}{n}I_{2,n} = \frac{1}{2}\left(\frac{2}{n}I_{\text{odd}} + \frac{2}{n}I_{\text{even}}\right),$$

where

$$I_{\text{odd}} := \#\{1 \le i \le n-1 | X_i = X_{i+1} = 1, i \text{ odd }\},$$
$$I_{\text{even}} := \#\{1 \le i \le n-1 | X_i = X_{i+1} = 1, i \text{ even }\}.$$

Then, $I_{\text{odd}} = \sum_{i=1,\text{ odd}}^{n-1} X_i \cdot X_{i+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor -1} Y_k$, where $Y_k = X_{2k+1} \cdot X_{2k+2}$ are iid Bern(1/4). Thus, by the strong law of large numbers

$$\frac{2}{n}I_{\text{odd}} = \frac{2}{n}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor -1} Y_k = \frac{2}{n}\cdot\left(\lfloor\frac{n}{2}\rfloor - 1\right)\cdot\bar{Y} \xrightarrow[n\to\infty]{a.s.} \mathbb{E}[Y_1] = \frac{1}{4}.$$

Similarly, one can show that

$$\frac{2}{n}I_{\text{even}} \xrightarrow[n\to\infty]{a.s.} \frac{1}{4}.$$

By Fact 1,

$$\frac{1}{n}I_{2,n} \xrightarrow[n\to\infty]{a.s.} \frac{1}{4},$$

thus $b_2 = \frac{1}{4}$.

Note: we could not apply the strong law of large numbers to the original $I_{2,n}$ since the latter is the sum of *non-independent* events.

3. For all $j = 0, 1, ..., k-1$ we define

$$I_j = \#\{1 \le i \le n-k+1 | X_i = X_{i+1} = ... = X_{i+k-1} = 1, i \text{ mod}_k = j\}.$$

Clearly,

$$\frac{1}{n}I_{k,n} = \frac{1}{k}\left(\frac{k}{n}\sum_{j=0}^{k-1} I_j\right).$$

Similarly as before, one can observe that for all $j = 0, 1, ..., k-1$

$$I_j = \sum_{i=1, i\text{mod}_k = j}^{n-1} \prod_{l=i}^{i+k} X_i = \sum_{h=0}^{\lfloor \frac{n}{k} \rfloor -1} Y_h,$$

where Y_h are iid Bern$(1/2^k)$. We can apply the strong law of large numbers and find that for $j = 0, 1, ..., k-1$,

$$\frac{k}{n}I_j \xrightarrow[n\to\infty]{a.s.} \frac{1}{2^k}.$$

By Fact 1, one can conclude that

$$\frac{1}{n} I_{k,n} \xrightarrow[n\to\infty]{a.s.} \frac{1}{2^k},$$

thus, $b_k = \frac{1}{2^k}$

4. a) Clearly,

$$\mathbb{P}(K_n \geq k) = \mathbb{P}(I_{k,n} \geq 1).$$

We can apply Markov's inequality and recall from point 3 that $\mathbb{E}[I_{k,n}] = \frac{n}{2^k}$, and find

$$\mathbb{P}(I_{k,n} \geq 1) \leq \mathbb{E}[I_{k,n}] = \frac{n}{2^k}.$$

b) Let $Y_m := \prod_{i=mk+1}^{(m+1)k} X_i$ for $m = 0, 1, ..., \lfloor \frac{n}{k} \rfloor - 1$. Note that Y_m are independent from each other.

$$\begin{aligned}
\mathbb{P}(K_n < k) &= \mathbb{P}(I_{k,n} = 0) \\
&\leq \mathbb{P}(\cap_{m=0}^{\lfloor \frac{n}{k} \rfloor - 1} \{Y_m = 0\}) \\
&= \prod_{m=0}^{\lfloor \frac{n}{k} \rfloor - 1} \mathbb{P}(\{Y_m = 0\}) \\
&= \left(1 - \frac{1}{2^k}\right)^{\lfloor \frac{n}{k} \rfloor}.
\end{aligned}$$

c) $\frac{\log 2}{\log n} K_n \xrightarrow[n\to\infty]{p} 1$ if and only if for all $\epsilon > 0$

$$\mathbb{P}(|\frac{\log 2}{\log n} K_n - 1| \geq \epsilon) \xrightarrow[n\to\infty]{} 0$$

Thus, let $\epsilon > 0$. We use point a) to show that

$$i) \quad \mathbb{P}(\frac{\log 2}{\log n} K_n \geq \epsilon + 1) \xrightarrow[n\to\infty]{} 0,$$

and point b) to show that

$$ii) \quad \mathbb{P}(\frac{\log 2}{\log n} K_n \leq 1 - \epsilon) \xrightarrow[n\to\infty]{} 0.$$

i) and ii) clearly imply the result.
Let us start with i)

$$\begin{aligned}
\mathbb{P}(\frac{\log 2}{\log n} K_n \geq \epsilon + 1) &= \mathbb{P}(K_n \geq (\epsilon + 1)\frac{\log n}{\log 2}) \\
&\leq \frac{n}{2^{\frac{\log n}{\log 2}(1+\epsilon)}} \\
&= \frac{1}{n^\epsilon} \xrightarrow[n\to\infty]{} 0, \qquad \forall \epsilon > 0,
\end{aligned}$$

where the inequality follows by applying a) with $k = (\epsilon + 1)\frac{\log n}{\log 2}$.

On the other hand, for ii),

$$\begin{aligned}
\mathbb{P}\left(\frac{\log 2}{\log n}K_n \le 1-\epsilon\right) &= \mathbb{P}\left(K_n \le (1-\epsilon)\frac{\log n}{\log 2}\right)\\
&\le \left(1-2^{-(1-\epsilon)\frac{\log n}{\log 2}}\right)^{\lfloor \frac{n}{(1-\epsilon)\frac{\log n}{\log 2}} \rfloor}\\
&= \left(1-n^{-(1-\epsilon)}\right)^{\lfloor \frac{n}{(1-\epsilon)\frac{\log n}{\log 2}} \rfloor}\\
&\le \exp\left(-n^{-(1-\epsilon)}\lfloor \frac{n}{(1-\epsilon)\frac{\log n}{\log 2}} \rfloor\right)\\
&\le \exp\left(-n^{-(1-\epsilon)}\left(\frac{n}{(1-\epsilon)\frac{\log n}{\log 2}}-1\right)\right)\\
&= \exp\left(-\left(\frac{n^{\epsilon}}{(1-\epsilon)\frac{\log n}{\log 2}}-o_n(1)\right)\right)\xrightarrow[n\to\infty]{} 0, \qquad \forall\epsilon>0,
\end{aligned}$$

where the first inequality follows by applying b) with $k=(\epsilon+1)\frac{\log n}{\log 2}$.

Exercise 7. Alice has 10 CHF and Bob has 5 CHF. They play a game. They repeatedly throw a fair coin (i.e. a coin such that $\mathbb{P}(\{\text{head}\})=\mathbb{P}(\{\text{tail}\})=1/2$). Each throw is independent from the others. When a tail comes up, Alice gives 1 CHF to Bob; and conversely when a head comes up, Bob gives 1 CHF to Alice. The first player that reaches 0 CHF loses, and the other player wins.

1. What is the probability that Alice wins?

2. What is the expected number of coin throws in a game?

Solution:

1. Let $T_k \in \{H,T\}$ be the output of the k-th coin throw, for $k=1,2,3,....$ We denote by $A(k)$ and $B(k)$ the budget of Alice and resp. Bob after the k-th coin throw ($A(0),B(0)$ being their initial budgets). Clearly for all k, $B(k)=15-A(k)$. Let

$$p(x)=\mathbb{P}(\text{Alice wins }|A(0)=x) \qquad x=0,1,2,...,15.$$

Clearly, $p(0)=0$ and $p(15)=1$. One can observe that for $x=1,2,...,14$

$$\begin{aligned}
p(x) &= \mathbb{P}(\text{Alice wins }|A(0)=x,A(1)=x-1)\mathbb{P}(T_1=T)+\mathbb{P}(\text{Alice wins }|A(0)=x,A(1)=x+1)\mathbb{P}(T_1=H)\\
&= \mathbb{P}(\text{Alice wins }|A(0)=x-1)\mathbb{P}(T_1=T)+\mathbb{P}(\text{Alice wins }|A(0)=x+1)\mathbb{P}(T_1=H)\\
&= \frac{1}{2}p(x-1)+\frac{1}{2}p(x+1).
\end{aligned}$$

Thus, $p(x)$ must satisfy the following conditions:

$$\begin{aligned}
&p(x)=\frac{p(x-1)+p(x+1)}{2}\\
&p(0)=0\\
&p(15)=1.
\end{aligned}$$

We can observe that the first condition implies that the function is linear as its rate of change is constant (since $p(x)-p(x-1)=p(x+1)-p(x)$). Then using $p(0)=0$ and $p(15)=1$, we conclude that $p(x)=\frac{1}{15}x$.

Thus, we conclude that $p(10)=2/3$.

2. Let C denote the number of coin throws. We denote by

$$e(x)=\mathbb{E}[C|A(0)=x].$$

Clearly, $e(0) = 0$ and $e(15) = 0$. One can observe that for $x = 1, 2, ..., 14$,

$$\begin{aligned} e(x) &= 1 + \mathbb{E}[C|A(0) = x, A(1) = x-1]\mathbb{P}(T_1 = T) + \mathbb{E}[C|A(0) = x, A(1) = x+1]\mathbb{P}(T_1 = H) \\ &= 1 + \mathbb{E}[C|A(0) = x-1]\mathbb{P}(T_1 = T) + \mathbb{E}[C|A(0) = x+1]\mathbb{P}(T_1 = H) \\ &= 1 + \frac{1}{2}e(x-1) + \frac{1}{2}e(x+1). \end{aligned}$$

Thus, $e(x)$ must satisfy the following conditions:

$$\begin{aligned} &e(x) = 1 + \frac{1}{2}e(x-1) + \frac{1}{2}e(x+1) \\ &e(0) = 0 \\ &e(15) = 0. \end{aligned}$$

One can check that these imply $e(x) = x(15-x)$. To see this, we can unroll the recursion starting from $e(0) = 0$:

$$\begin{aligned} &e(2) = 2(e(1)+1) - 4 \\ &e(3) = 3(e(1)+1) - 9 \\ &e(4) = 4(e(1)+1) - 16 \\ &... \end{aligned}$$

This suggests the general formula $e(x) = x(e(1)+1) - x^2$. The condition $e(15) = 0$ allows us to calculate $e(1) = 14$ which leads to the formula $e(x) = x(15-x)$ that satisfies all the three conditions on $e(x)$. Thus, $e(10) = 10 \cdot (15-10) = 50$.